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The Determinants of Matrices Constructed by Subdiagonal, Main Diagonal and Superdiagonal

N. Mirashe[1], A. R. Moghaddamfar[2,3*], and S. H. Mozafari[1]

(Submitted by O.E. Tikhonov)

[1] *Department of Computer Hardware Engineering, Faculty of Electrical Engineering, K. N. Toosi University of Technology, Tehran, Iran*

[2] *Department of Mathematics, Faculty of Science, K. N. Toosi University of Technology, P. O. Box 16315-1618, Tehran, Iran*

[3] *Research Institute for Fundamental Sciences (RIFS), Tabriz, Iran*

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Abstract—The purpose of this article is to prove several evaluations of determinants of matrices, the entries of which are given by the recurrences

$$a_{i,j} = \begin{cases} a_{i,j-2} + a_{i+1,j-1} + a_{i+2,j}, & \text{if} \quad j \geq i+2; \\ a_{i-2,j} + a_{i-1,j+1} + a_{i,j+2}, & \text{if} \quad i \geq j+2, \end{cases}$$

with various choices for main diagonal $a_{i,i}$, superdiagonal $a_{i,i+1}$ and subdiagonal $a_{i+1,i}$.

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1. INTRODUCTION

Much remarkable attention has been paid to the symbolic evaluation of determinants in literature (for instance, see [2–11]). Certain practical and efficient tools so as to evaluate determinants are provided in [2] and [3]. They include many determinants explicitly evaluated which appeared in the literature. They also put forward several references where further formulae can be found.

Bacher ([1], Section 8) introduces the *diagonal construction* as follows. Let $\gamma = (\gamma_i)_{i\geq 1}$ be a sequence and let u_1, u_2, l_1, l_2 be four constants. Then the diagonal-construction is the (infinite) matrix $D_\gamma^{(u_1,u_2,l_1,l_2)}$ with entries

$$d_{i,j} = \begin{cases} \gamma_i, & \text{if} \quad i = j \geq 1; \\ u_1 d_{i,j-1} + u_2 d_{i+1,j}, & \text{if} \quad j > i \geq 1, \\ l_1 d_{i-1,j} + l_2 d_{i,j+1}, & \text{if} \quad i > j \geq 1. \end{cases}$$

Now it is easy to see that, we have

$$d_{i,j} = u_1^2 d_{i,j-2} + 2u_1u_2 d_{i+1,j-1} + u_2^2 d_{i+2,j} \quad \text{for} \quad j \geq i+2,$$

and also for $i \geq j+2$, we have

$$d_{i,j} = l_1^2 d_{i-2,j} + 2l_1l_2 d_{i-1,j+1} + l_2^2 d_{i,j+2}.$$

Through inspiring from the above relations, we attempt to put forward a new method for constructing certain matrices. Our way of construction is based on considering three specific sequences for the entries

*E-mail: moghadam@kntu.ac.ir;moghadam@mail.ipm.ir
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on main diagonal, superdiagonal and subdiagonal. Then, using some recurrence relations the other entries are constructed. Let us present this sort of constructing formally.

Consider the $n \times n$ matrix $A = (A_{i,j})_{1\le i,j\le n}$. Suppose $\alpha = (\alpha_i)_{i\ge 1}$, $\beta = (\beta_i)_{i\ge 1}$ and $\gamma = (\gamma_i)_{i\ge 1}$ be three certain sequences. We put $A_{i,i+1} = \alpha_i$, $A_{i,i} = \beta_i$ and $A_{i+1,i} = \gamma_i$. Then, using the following recurrence relations:

$$A_{i,j} = \begin{cases} A_{i-2,j} + A_{i-1,j+1} + A_{i,j+2}, & \text{if} \quad i \ge j+2; \\ A_{i,j-2} + A_{i+1,j-1} + A_{i+2,j}, & \text{if} \quad j \ge i+2, \end{cases}$$

we construct the other entries.

In fact, our main aim in this article is to choose three sequences, say, α, β and γ and construct the $n \times n$ matrix A as mentioned above. Then, we evaluate the determinant of A.

Our major tools for the evaluation are the LU-decomposition of matrices and elementary row and column operations. In linear algebra, the LU-decomposition is a matrix decomposition which writes a matrix as the product of a lower and upper triangular matrix (see [3]). This decomposition is used in a numerical analysis to solve the systems of linear equations or find the inverse of a matrix. Indeed, the LU-decomposition is illustrated as follows: Let A be a square matrix. An LU-decomposition is a decomposition of the form $A = LU$, where L and U are lower and upper triangular matrices (of the same size), respectively.

The notation used in this article are standard and they could be found in most of Linear Algebra textbooks. However, we use the notation A^T for the transpose of A. Also, we denote by $\mathrm{R}_i(A)$ and $\mathrm{C}_j(A)$ the row i and the column j of A, respectively. Note that, when $m > n$, we assume that $\prod_{j=m}^{n} b_j = 1$. We denote the elementary row and column operations of type three by $O_{ij}(\lambda)$ and $O'_{ij}(\lambda)$, respectively, where $i \ne j$ and λ a scalar. Therefore, we have

$$\mathrm{R}_k(O_{ij}(\lambda)A) = \begin{cases} \mathrm{R}_i(A) + \lambda \mathrm{R}_j(A), & \text{if} \quad k = i; \\ \mathrm{R}_k(A), & \text{if} \quad k \ne i, \end{cases}$$

and

$$\mathrm{C}_k(AO'_{ij}(\lambda)) = \begin{cases} \mathrm{C}_i(A) + \lambda \mathrm{C}_j(A), & \text{if} \quad k = i; \\ \mathrm{C}_k(A), & \text{if} \quad k \ne i, \end{cases}$$

Given $G(1)$ and $G(2)$, a *Gibonacci sequence* (or *generalized Fibonacci sequence*) $\{G(m)\}_{m=1}^{\infty}$ is recursively defined by $G(m+2) = G(m+1) + G(m)$, for $m \ge 1$. A Toeplitz matrix is an $n \times n$ matrix $T_n = (t_{i,j})$ where $t_{i,j} = t_{j-i}$, i.e., a matrix of the form

$$T_n = \begin{bmatrix} t_0 & t_1 & t_2 & \dots & t_{n-1} \\ t_{-1} & t_0 & t_1 & \dots & t_{n-2} \\ t_{-2} & t_{-1} & t_0 & \dots & t_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{1-n} & t_{2-n} & t_{3-n} & \dots & t_0 \end{bmatrix}.$$

The goal of this paper is to prove the following theorem:

Main Theorem *Let $(A_{i,j})_{i,j\ge 1}$ be the doubly indexed sequence given by the recurrences*

$$A_{i,j} = \begin{cases} A_{i-2,j} + A_{i-1,j+1} + A_{i,j+2}, & \text{if} \quad i \ge j+2; \\ A_{i,j-2} + A_{i+1,j-1} + A_{i+2,j}, & \text{if} \quad j \ge i+2. \end{cases}$$

Then:

(1) if $A_{i,i} = G(i)$ for $1 \le i \le n$ and $A_{i,i+1} = A_{i+1,i} = 0$ for $1 \le i \le n-1$, then we have

$$\det(A_{i,j})_{1\le i,j\le n} = \begin{cases} G(1), & \text{if } n = 1; \\ (-1)^{n-2} G(n-1) G(n) \prod_{i=3}^{n} [G(i) + G(i+2)], & \text{if } n \ge 2\,. \end{cases}$$

(2) if $A_{i,i}=0$ for $1\le i\le 2n$ and $A_{i,i+1}=A_{i+1,i}=G(i)$ for $1\le i\le 2n-1$, then we have

$$\det(A_{i,j})_{1\le i,j\le 2n}=\begin{cases}-G(1)^2, & \text{if } n=1;\\ (-1)^n\Big\{\omega G(2n-1)\prod_{i=1}^{n-2}[G(2i)+G(2i+2)]\Big\}^2, & \text{if } n\ge 2\,,\end{cases}$$

where $\omega=G(1)-2G(2)$

(3) if $A_{i,i}=q^{i-1}$ for $1\le i\le n$ and $A_{i,i+1}=A_{i+1,i}=0$ for $1\le i\le n-1$, then we have

$$\det(A_{i,j})_{1\le i,j\le n}=\begin{cases}q^{n-1}, & \text{if}\quad n\le 2;\\ q^{\binom{n-2}{2}+1}(-\lambda)^{n-2}, & \text{if}\quad n\ge 3,\end{cases}$$

where $\lambda=(1+q)^2(1+q^2)$.

(4) if $A_{i,i}=0$ for $1\le i\le 2n$, $A_{i,i+1}=aq^{i-1}$, $A_{i+1,i}=bq^{i-1}$ for $1\le i\le 2n-1$, then we have

$$\det(A_{i,j})_{1\le i,j\le 2n}=(-1)^n abq^{2(n-1)^2}\big[(aq-b\mu)(bq-a\mu)\big]^{n-1},$$

where $\mu=1+q+q^2$.

(5) if $A_{i,i}=0$ for $1\le i\le 2n$ and $A_{i,i+1}=-A_{i+1,i}=\alpha_i$ for $1\le i\le 2n-1$, then we have

$$\det(A_{i,j})_{1\le i,j\le 2n}=2^{2(n-3)}(\alpha_1+\alpha_2)^{2(n-1)}\big[4\alpha_1+(n-1)(n-4)(\alpha_1-\alpha_2)\big]^2.$$

where $\{\alpha_i\}_{i=1}^{\infty}$ is a 2-periodic sequence.

In the next section, we will prove the Main Theorem in separate theorems.

2. PROOFS

Theorem 1. *Let $\{G(i)\}_{i=1}^{\infty}$ be a Gibonacci sequence. Let $(A_{i,j})_{1\le i,j\le n}$ be the doubly indexed sequence given by the recurrences*

$$A_{i,j}=\begin{cases}A_{i-2,j}+A_{i-1,j+1}+A_{i,j+2}, & \text{if}\quad i\ge j+2;\\ A_{i,j-2}+A_{i+1,j-1}+A_{i+2,j}, & \text{if}\quad j\ge i+2,\end{cases}$$

and the initial conditions $A_{i,i}=G(i)$ for $1\le i\le n$ and $A_{i,i+1}=A_{i+1,i}=0$ for $1\le i\le n-1$. We set $d(n)=\det(A_{i,j})_{1\le i,j\le n}$. Then the sequence $\{d(n)\}_{n=1}^{\infty}$ satisfies the following equalities

$$d(1)=G(1),\quad d(2)=G(2)d(1),$$

and

$$d(n)=-\frac{G(n)}{G(n-2)}\big[G(n)+G(n+2)\big]d(n-1),\quad \text{for}\quad n\ge 3.$$

Furthermore, an explicit formula for $d(n)$ is as follows

$$d(n)=\begin{cases}G(1), & \text{if } n=1;\\ (-1)^{n-2}G(n-1)G(n)\prod_{i=3}^{n}[G(i)+G(i+2)], & \text{if } n\ge 2\,.\end{cases}$$

Proof. Let A denote the matrix $(A_{i,j})_{1\le i,j\le n}$. Note that, we can determine the entries $A_{i,j}$ as follows:

$$A_{i,j}=\begin{cases}2^{\frac{i-j}{2}}G(i), & 0\le i-j=\text{even number};\\ 2^{\frac{j-i}{2}}G(j), & 0\le j-i=\text{even number};\\ 0, & \text{otherwise}.\end{cases}$$

We apply LU-decomposition method. We claim that

$$A=L\cdot U,$$

where $L=(L_{i,j})_{1\le i,j\le n}$ is a lower triangular matrix with the coefficients

$$L_{i,j}=\begin{cases}\dfrac{2^{\frac{i-j}{2}}G(i)}{G(j)}, & 0\le i-j=\text{even number};\\ 0, & \text{otherwise},\end{cases}$$

and $U=(U_{i,j})_{1\le i,j\le n}$ is an upper triangular matrix with the coefficients

$$U_{i,j}=\begin{cases}2^{\frac{j-i}{2}}G(j), & 1\le i\le 2, 0\le j-i=\text{even number};\\ -2^{\frac{j-i}{2}}\dfrac{G(j)}{G(i-2)}[G(i)+G(i+2)], & 3\le i\le n, 0\le j-i=\text{even number};\\ 0, & \text{otherwise}.\end{cases}$$

For instance, when $n=5$, the matrices A, L and U are given by

$$A=\begin{bmatrix}G(1) & 0 & 2G(3) & 0 & 2^2G(5)\\ 0 & G(2) & 0 & 2G(4) & 0\\ 2G(3) & 0 & G(3) & 0 & 2G(5)\\ 0 & 2G(4) & 0 & G(4) & 0\\ 2^2G(5) & 0 & 2G(5) & 0 & G(5)\end{bmatrix},\quad L=\begin{bmatrix}1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ \frac{2G(3)}{G(1)} & 0 & 1 & 0 & 0\\ 0 & \frac{2G(4)}{G(2)} & 0 & 1 & 0\\ \frac{2^2G(5)}{G(1)} & 0 & \frac{2G(5)}{G(3)} & 0 & 1\end{bmatrix},$$

$$U=\begin{bmatrix}G(1) & 0 & 2G(3) & 0 & 2^2G(5)\\ 0 & G(2) & 0 & 2G(4) & 0\\ 0 & 0 & -\frac{G(3)}{G(1)}[G(3)+G(5)] & 0 & -2\frac{G(5)}{G(1)}[G(3)+G(5)]\\ 0 & 0 & 0 & -\frac{G(4)}{G(2)}[G(4)+G(6)] & 0\\ 0 & 0 & 0 & 0 & -\frac{G(5)}{G(3)}[G(5)+G(7)]\end{bmatrix}.$$

The matrix L is a lower triangular matrix with 1's on the diagonal, whereas U is an upper triangular matrix with diagonal entries $U_{1,1}=G(1)$, $U_{2,2}=G(2)$ and

$$U_{i,i}=-\frac{G(i)}{G(i-2)}\big[G(i)+G(i+2)\big],\qquad i\ge 3.$$

It is obvious that the claim immediately implies the theorem.

We verify the claim by a direct calculation. In order to prove the claim we need only to show that

$$(L\cdot U)_{i,j}=A_{i,j},\quad \text{when } i+j \text{ is an even number.}$$

Similar to the proof of Theorem 1, we distinguish two cases separately:

Case 1. i,j odd. In this case we have

$$\mathrm{R}_i(L)=\left(\frac{2^{\frac{i-1}{2}}G(i)}{G(1)},0,\frac{2^{\frac{i-3}{2}}G(i)}{G(3)},0,\frac{2^{\frac{i-5}{2}}G(i)}{G(5)},0,\dots,\frac{2^{\frac{i-j}{2}}G(i)}{G(j)},0,\dots,\frac{2^{\frac{i-i}{2}}G(i)}{G(i)}=1,0,0,\dots,0\right),$$

and

$$\mathrm{C}_j(U)=\Bigg(2^{\frac{j-1}{2}}G(j),0,-2^{\frac{j-3}{2}}\frac{G(j)}{G(1)}[G(3)+G(5)],0,-2^{\frac{j-5}{2}}\frac{G(j)}{G(3)}[G(5)+G(7)],\dots,$$
$$0,-2^{\frac{j-i}{2}}\frac{G(j)}{G(i-2)}[G(i)+G(i+2)],0,\dots,-\frac{G(j)}{G(j-2)}[G(j)+G(j+2)],0,0,\dots,0\Bigg)^T.$$

Assume first that $i \geq j$. For $i = 1$, the result is straightforward. Therefore, we may assume that $i \geq 3$. In this case, we obtain

$$(L \cdot U)_{i,j} = \sum_{k=1}^{j} L_{i,k} U_{k,j} = \frac{2^{\frac{i+j-2}{2}} G(i) G(j)}{G(1)}$$
$$+ \sum_{l=0}^{\frac{j-3}{2}} \frac{2^{\frac{i-(2l+3)}{2}} G(i)}{G(2l+3)} \frac{-2^{\frac{j-(2l+3)}{2}} G(j)[G(2l+3) + G(2l+5)]}{G(2l+1)}$$
$$= \frac{2^{\frac{i+j-2}{2}} G(i) G(j)}{G(1)} - G(i)G(j) \sum_{l=0}^{\frac{j-3}{2}} 2^{\frac{i+j-2(2l+3)}{2}} \frac{[4G(2l+3) - G(2l+1)]}{G(2l+3)G(2l+1)}$$
$$= 2^{\frac{i-j}{2}} G(i) \left\{ \frac{2^{\frac{j-1}{2}} G(j)}{G(1)} - G(j) \sum_{l=0}^{\frac{j-3}{2}} 2^{j-(2l+3)} \left[\frac{4}{G(2l+1)} - \frac{1}{G(2l+3)} \right] \right\}$$
$$= 2^{\frac{i-j}{2}} G(i) \left\{ \frac{2^{\frac{j-1}{2}} G(j)}{G(1)} - \frac{2^{\frac{j-1}{2}} G(j)}{G(1)} + 1 \right\} = 2^{\frac{i-j}{2}} G(i) = A_{i,j}.$$

The case when $i \leq j$ is similar.

Case 2. i, j even. In this case we have

$$\mathrm{R}_i(L) = \left(0, \frac{2^{\frac{i-2}{2}} G(i)}{G(2)}, 0, \frac{2^{\frac{i-4}{2}} G(i)}{G(4)}, 0, \frac{2^{\frac{i-6}{2}} G(i)}{G(6)}, \ldots, 0, \frac{2^{\frac{i-j}{2}} G(i)}{G(j)}, \ldots, 0, \frac{2^{\frac{i-i}{2}} G(i)}{G(i)} = 1, 0, 0, \ldots, 0 \right),$$

and

$$\mathrm{C}_j(U) = \Bigg(0, 2^{\frac{j-2}{2}} G(j), 0, -2^{\frac{j-4}{2}} \frac{G(j)}{G(2)}[G(4) + G(6)], 0, -2^{\frac{j-6}{2}} \frac{G(j)}{G(4)}[G(6) + G(8)], \ldots,$$
$$0, -2^{\frac{j-i}{2}} \frac{G(j)}{G(i-2)}[G(i) + G(i+2)], \ldots, 0, -\frac{G(j)}{G(j-2)}[G(j) + G(j+2)], 0, 0, \ldots, 0 \Bigg)^T.$$

Suppose that $i \geq j$. For $i = 2$, the result is straightforward again. Therefore, we may assume that $i \geq 4$. Similar to the previous case, we obtain

$$(L \cdot U)_{i,j} = \sum_{k=1}^{j} L_{i,k} U_{k,j} = \frac{2^{\frac{i+j-4}{2}} G(i) G(j)}{G(2)}$$
$$+ \sum_{l=2}^{\frac{j}{2}} \frac{2^{\frac{i-2l}{2}} G(i)}{G(2l)} \frac{-2^{\frac{j-2l}{2}} G(j)[G(2l) + G(2l+2)]}{G(2l-2)}$$
$$= \frac{2^{\frac{i+j-4}{2}} G(i) G(j)}{G(2)} - G(i)G(j) \sum_{l=2}^{\frac{j}{2}} 2^{\frac{i+j-4l}{2}} \frac{[4G(2l) - G(2l-2)]}{G(2l-2)G(2l)}$$
$$= 2^{\frac{i-j}{2}} G(i) \left\{ \frac{2^{j-2} G(j)}{G(2)} - G(j) \sum_{l=2}^{\frac{j}{2}} 2^{j-2l} \left[\frac{4}{G(2l-2)} - \frac{1}{G(2l)} \right] \right\}$$
$$= 2^{\frac{i-j}{2}} G(i) \left\{ \frac{2^{j-2} G(j)}{G(2)} - \frac{2^{\frac{j-2}{2}} G(j)}{G(2)} + 1 \right\} = 2^{\frac{i-j}{2}} G(i) = A_{i,j}.$$

The case when $j \geq i$ is similar. The proof is complete. □

Theorem 2. *Let $\{G(i)\}_{i=1}^{\infty}$ be a Gibonacci sequence. Let $(A_{i,j})_{1\le i,j\le 2n}$ be the doubly indexed sequence given by the recurrences*

$$A_{i,j}=\begin{cases}A_{i-2,j}+A_{i-1,j+1}+A_{i,j+2}, & \text{if} \quad i\ge j+2;\\ A_{i,j-2}+A_{i+1,j-1}+A_{i+2,j}, & \text{if} \quad j\ge i+2,\end{cases}$$

and the initial conditions $A_{i,i}=0$ for $1\le i\le 2n$ and $A_{i,i+1}=A_{i+1,i}=G(i)$ for $1\le i\le 2n-1$. Then we have

$$\det(A_{i,j})_{1\le i,j\le 2n}=\begin{cases}-G(1)^2, & \text{if } n=1;\\ (-1)^n\left\{G(2n-1)\Big(G(1)-2G(2)\Big)\prod_{i=1}^{n-2}[G(2i)+G(2i+2)]\right\}^2, & \text{if } n\ge 2.\end{cases}$$

Proof. If $n=1$, the result is straightforward. Therefore, we may assume that $n\ge 2$. Let A denote the matrix $(A_{i,j})_{1\le i,j\le 2n}$. Note that, we can determine the entries $A_{i,j}$ as follows:

$$A_{i,j}=\begin{cases}2^{\frac{i-j-1}{2}}G(i-1), & \text{if } i-j \text{ is an odd number;}\\ 2^{\frac{j-i-1}{2}}G(j-1), & \text{if } j-i \text{ is an odd number,}\\ 0, & \text{otherwise.}\end{cases}$$

For instance, when $n=3$, the matrix A is given by

$$A=\begin{bmatrix}0 & G(1) & 0 & 2G(3) & 0 & 2^2G(5)\\ G(1) & 0 & G(2) & 0 & 2G(4) & 0\\ 0 & G(2) & 0 & G(3) & 0 & 2G(5)\\ 2G(3) & 0 & G(3) & 0 & G(4) & 0\\ 0 & 2G(4) & 0 & G(4) & 0 & G(5)\\ 2^2G(5) & 0 & 2G(5) & 0 & G(5) & 0\end{bmatrix}.$$

Now we apply the row and column operations. Through changing the rows and columns such that the columns with nonzero first entries are put together and then the rows with zero first entries are beside one another, we can deduce that

$$\det(A)=(-1)^n\det\begin{bmatrix}B & 0\\ 0 & B^T\end{bmatrix},$$

where $B=(B_{i,j})_{1\le i,j\le n}$ is an $n\times n$ matrix with entries:

$$B_{i,j}=\begin{cases}2^{j-i}G(2j-1), & \text{if} \quad i<j;\\ G(2i-1), & \text{if} \quad i=j;\\ 2^{i-j-1}G(2i-2) & \text{if} \quad i>j.\end{cases}$$

Again, we apply the following row and column operations:

$$B^*=\prod_{k=2}^{n}O_{k-1,k}(-2)B.$$

In fact, B^* is a lower triangular matrix with diagonal entries:

$$G(1)-2G(2), G(3)-2G(4),\ldots,G(2i-1)-2G(2i),\ldots,G(2n-3)-2G(2n-2),G(2n-1),$$

or equivalently

$$G(1)-2G(2),-(G(2)+G(4)),\ldots,-(G(2i-2)+G(2i)),\ldots,-(G(2n-4)+G(2n-2)),G(2n-1).$$

Therefore, we obtain

$$\det(B)=\det(B^*)=(-1)^{n-2}G(2n-1)\Big(G(1)-2G(2)\Big)\prod_{i=1}^{n-2}[G(2i)+G(2i+2)],$$

and so

$$\det(A)=(-1)^n\left\{G(2n-1)\Big(G(1)-2G(2)\Big)\prod_{i=1}^{n-2}[G(2i)+G(2i+2)]\right\}^2,$$

as desired. □

Theorem 3. *Let $(A_{i,j})_{1\le i,j\le n}$ be the doubly indexed sequence given by the recurrences*

$$A_{i,j}=\begin{cases}A_{i-2,j}+A_{i-1,j+1}+A_{i,j+2}, & \text{if } \ i\ge j+2;\\ A_{i,j-2}+A_{i+1,j-1}+A_{i+2,j}, & \text{if } \ j\ge i+2,\end{cases}$$

and the initial conditions $A_{i,i}=q^{i-1}$ for $1\le i\le n$ and $A_{i,i+1}=A_{i+1,i}=0$ for $1\le i\le n-1$. Then we have

$$\det(A_{i,j})_{1\le i,j\le n}=\begin{cases}q^{n-1}, & \text{if } \ n\le 2;\\ q^{\binom{n-2}{2}+1}(-\lambda)^{n-2}, & \text{if } \ n\ge 3,\end{cases}$$

where $\lambda=(1+q)^2(1+q^2)$.

Proof. For convenience, we put $\lambda=(1+q)^2(1+q^2)$ and $\mu=1+q+q^2$. An easy computation shows that

$$\mu^2-\lambda=q^2. \tag{1}$$

Let A denote the matrix $(A_{i,j})_{1\le i,j\le n}$. It is worth noticing that the entries of A can be obtained as follows:

$$A_{i,j}=\begin{cases}q^{j-1}\mu^{\frac{i-j}{2}}, & \text{if } \ i-j \ \text{ is an even number;}\\ q^{i-1}\mu^{\frac{j-i}{2}}, & \text{if } \ j-i \ \text{ is an even number,}\\ 0, & \text{otherwise,}\end{cases}$$

Again, we apply LU-decomposition method. We claim that

$$A=L\cdot U,$$

where $L=(L_{i,j})_{1\le i,j\le n}$ is a lower triangular matrix with the coefficients

$$L_{i,j}=\begin{cases}\mu^{\frac{i-j}{2}}, & \text{if } \ i-j \ \text{ is an even number;}\\ 0, & \text{otherwise,}\end{cases}$$

and $U=(U_{i,j})_{1\le i,j\le n}$ is an upper triangular matrix with the coefficients

$$U_{i,j}=\begin{cases}q^{i-1}\mu^{\frac{j-i}{2}}, & \text{if } \ 1\le i\le 2 \ \text{ and } \ j-i \text{ is an even number;}\\ -\lambda q^{i-3}\mu^{\frac{j-i}{2}}, & \text{if } \ 3\le i\le n \ \text{ and } \ j-i \ \text{ is an even number,}\\ 0, & \text{otherwise.}\end{cases}$$

For instance, when $n=6$, the matrices A, L and U are given by

$$A=\begin{bmatrix}1 & 0 & \mu & 0 & \mu^2 & 0\\ 0 & q & 0 & q\mu & 0 & q\mu^2\\ \mu & 0 & q^2 & 0 & q^2\mu & 0\\ 0 & q\mu & 0 & q^3 & 0 & q^3\mu\\ \mu^2 & 0 & q^2\mu & 0 & q^4 & 0\\ 0 & q\mu^2 & 0 & q^3\mu & 0 & q^5\end{bmatrix},\quad L=\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\ \mu&0&1&0&0&0\\0&\mu&0&1&0&0\\ \mu^2&0&\mu&0&1&0\\0&\mu^2&0&\mu&0&1\end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 0 & \mu & 0 & \mu^2 & 0 \\ 0 & q & 0 & q\mu & 0 & q^2\mu^2 \\ 0 & 0 & -\lambda & 0 & -\lambda\mu & 0 \\ 0 & 0 & 0 & -\lambda q & 0 & -\lambda q\mu \\ 0 & 0 & 0 & 0 & -\lambda q^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\lambda q^3 \end{bmatrix}.$$

Note that, by the structure of matrices A, L and U, *the* (i,j)*-entry of these matrices is zero if* $i+j$ *is an odd number*. Moreover, the matrix L is a lower triangular matrix with 1's on the diagonal, whereas U is an upper triangular matrix with diagonal entries $U_{1,1}=1$, $U_{2,2}=q$ and

$$U_{i,i} = -\lambda q^{i-3}, \qquad i \geq 3.$$

It is obvious that the claimed decomposition of A immediately implies the validity of the theorem.

For the proof of the claim we compute the (i,j)-entry of $L\cdot U$, by definition, it is $\sum_{k=1}^{n} L_{i,k}U_{k,j}$, it is easy to see that, if $i+j$ is an odd number, then $(L\cdot U)_{i,j} = \sum_{k=1}^{n} L_{i,k}U_{k,j} = 0 = A_{i,j}$, because for any k either $i+k$ or $k+j$ is odd. Therefore, in order to prove the claim we need only show that

$$(L\cdot U)_{i,j} = A_{i,j}, \quad \text{when} \quad i+j \quad \text{is an even number.}$$

We distinguish two cases separately:

Case 1. i,j odd. In this case we have

$$\mathrm{R}_i(L) = (\mu^{\frac{i-1}{2}}, 0, \mu^{\frac{i-3}{2}}, 0, \mu^{\frac{i-5}{2}}, 0, \ldots, \mu^{\frac{i-i}{2}}, 0, 0, \ldots, 0)$$

and

$$\mathrm{C}_j(U) = \left(\mu^{\frac{j-1}{2}}, 0, -\lambda\mu^{\frac{j-3}{2}}, 0, -\lambda q^2\mu^{\frac{j-5}{2}}, 0, -\lambda q^4\mu^{\frac{j-7}{2}}, 0, \ldots, -\lambda q^{j-3}\mu^{\frac{j-j}{2}}, 0, 0, \ldots, 0\right)^T.$$

Assume first that $i \geq j$. For $i=1$, the result is straightforward. Therefore, we may assume that $i \geq 3$. Hence, we obtain

$$(L\cdot U)_{i,j} = \sum_{k=1}^{j} L_{i,k}U_{k,j} = \mu^{\frac{i+j-2}{2}} - \lambda\sum_{l=0}^{\frac{j-3}{2}} q^{2l}\mu^{\frac{i+j-2(2l+3)}{2}} = \mu^{\frac{i+j-6}{2}}(\mu^2-\lambda) - \lambda\sum_{l=1}^{\frac{j-3}{2}} q^{2l}\mu^{\frac{i+j-2(2l+3)}{2}}$$

$$= \mu^{\frac{i+j-6}{2}}q^2 - \lambda\sum_{l=1}^{\frac{j-3}{2}} q^{2l}\mu^{\frac{i+j-2(2l+3)}{2}} \quad \text{(by Eq. (1))}$$

$$= \mu^{\frac{i+j-10}{2}}q^2(\mu^2-\lambda) - \lambda\sum_{l=2}^{\frac{j-3}{2}} q^{2l}\mu^{\frac{i+j-2(2l+3)}{2}} = \mu^{\frac{i+j-10}{2}}q^4 - \lambda\sum_{l=2}^{\frac{j-3}{2}} q^{2l}\mu^{\frac{i+j-2(2l+3)}{2}} \quad \text{(by Eq. (1))}$$

$$\vdots$$

$$= \mu^{\frac{i-j+4}{2}}q^{j-3} - \lambda\sum_{l=\frac{j-3}{2}}^{\frac{j-3}{2}} q^{2l}\mu^{\frac{i+j-2(2l+3)}{2}} = \mu^{\frac{i-j}{2}}(\mu^2-\lambda)q^{j-3} = \mu^{\frac{i-j}{2}}q^{j-1} = A_{i,j}.$$

The case when $i \leq j$ is similar.

Case 2. i,j even. In this case we have

$$\mathrm{R}_i(L) = (0, \mu^{\frac{i-2}{2}}, 0, \mu^{\frac{i-4}{2}}, 0, \mu^{\frac{i-6}{2}}, 0, \ldots, \mu^{\frac{i-i}{2}}, 0, 0, \ldots, 0)$$

and

$$\mathrm{C}_j(U) = \left(0, q\mu^{\frac{j-2}{2}}, 0, -\lambda q\mu^{\frac{j-4}{2}}, 0, -\lambda q^3\mu^{\frac{j-6}{2}}, 0, -\lambda q^5\mu^{\frac{j-8}{2}}, 0, \ldots, -\lambda q^{j-3}\mu^{\frac{j-j}{2}}, 0, 0, \ldots, 0\right)^T.$$

Suppose that $i \geq j$. For $i = 2$, the result is straightforward again. Therefore, we may assume that $i \geq 4$. Similar to the previous case, we obtain

$$(L \cdot U)_{i,j} = \sum_{k=1}^{j} L_{i,k}U_{k,j} = q\mu^{\frac{i+j-4}{2}} - \lambda \sum_{l=0}^{\frac{j-4}{2}} q^{2l+1}\mu^{\frac{i+j-2(2l+4)}{2}} = q\mu^{\frac{i+j-8}{2}}(\mu^2 - \lambda)$$

$$- \lambda \sum_{l=1}^{\frac{j-4}{2}} q^{2l+1}\mu^{\frac{i+j-2(2l+4)}{2}} = q^3\mu^{\frac{i+j-8}{2}} - \lambda \sum_{l=1}^{\frac{j-4}{2}} q^{2l+1}\mu^{\frac{i+j-2(2l+4)}{2}} \quad \text{(by Eq. (1))}$$

$$= q^3\mu^{\frac{i+j-12}{2}}(\mu^2 - \lambda) - \lambda \sum_{l=2}^{\frac{j-4}{2}} q^{2l+1}\mu^{\frac{i+j-2(2l+4)}{2}} = q^5\mu^{\frac{i+j-12}{2}} - \lambda \sum_{l=2}^{\frac{j-4}{2}} q^{2l+1}\mu^{\frac{i+j-2(2l+4)}{2}} \quad \text{(by Eq. (1))}$$

$$\vdots$$

$$= q^{j-3}\mu^{\frac{i-j+4}{2}} - \lambda \sum_{l=\frac{j-4}{2}}^{\frac{j-4}{2}} q^{2l+1}\mu^{\frac{i+j-2(2l+4)}{2}} = q^{j-3}\mu^{\frac{i-j}{2}}(\mu^2 - \lambda) = q^{j-1}\mu^{\frac{i-j}{2}} = A_{i,j}.$$

The case when $j \geq i$ is similar. The proof is complete. □

Theorem 4. *Let $(A_{i,j})_{1 \leq i,j \leq 2n}$ be the doubly indexed sequence given by the recurrences*

$$A_{i,j} = \begin{cases} A_{i-2,j} + A_{i-1,j+1} + A_{i,j+2}, & \text{if} \quad i \geq j+2; \\ A_{i,j-2} + A_{i+1,j-1} + A_{i+2,j}, & \text{if} \quad j \geq i+2, \end{cases}$$

and the initial conditions $A_{i,i} = 0$ for $1 \leq i \leq 2n$ and $A_{i,i+1} = aq^{i-1}$, $A_{i+1,i} = bq^{i-1}$ for $1 \leq i \leq 2n-1$. Then we have

$$\det(A_{i,j})_{1 \leq i,j \leq 2n} = (-1)^n abq^{2(n-1)^2}\left[(aq - b\mu)(bq - a\mu)\right]^{n-1},$$

where $\mu = 1 + q + q^2$.

Proof. Let A denote the matrix $(A_{i,j})_{1 \leq i,j \leq 2n}$. Then, we have

$$A_{i,j} = \begin{cases} bq^{j-1}\mu^{\frac{i-j-1}{2}}, & \text{if } i-j \text{ is an odd number;} \\ aq^{i-1}\mu^{\frac{j-i-1}{2}}, & \text{if } j-i \text{ is an odd number,} \\ 0, & \text{otherwise.} \end{cases}$$

For instance, when $n = 3$, the matrix A is given by

$$A = \begin{bmatrix} 0 & a & 0 & a\mu & 0 & a\mu^2 \\ b & 0 & aq & 0 & aq\mu & 0 \\ 0 & bq & 0 & aq^2 & 0 & aq^2\mu \\ b\mu & 0 & bq^2 & 0 & aq^3 & 0 \\ 0 & bq\mu & 0 & bq^3 & 0 & aq^4 \\ b\mu^2 & 0 & bq^2\mu & 0 & bq^4 & 0 \end{bmatrix}.$$

If $a = 0$ or $b = 0$, then the result is obvious. Therefore, we may assume that $a, b \neq 0$. Now we apply the following row and column operations:

$$A_1 = \left(\prod_{k=2}^{n} O_{2k,2}(-\mu^{k-1})\right) A \left(\prod_{k=2}^{n} O'_{2k,2}(-\mu^{k-1})\right),$$

$$A_2 = \left(\prod_{k=3}^{n} O_{2k,4}(-\mu^{k-1})\right) A_1 \left(\prod_{k=3}^{n} O'_{2k,4}(-\mu^{k-1})\right),$$

$$A_3 = \left(\prod_{k=4}^{n} O_{2k,6}(-\mu^{k-1})\right) A_2 \left(\prod_{k=4}^{n} O'_{2k,6}(-\mu^{k-1})\right),$$

$$\vdots$$

$$A_{n-1} = \left(\prod_{k=n}^{n} O_{2k,2n-2}(-\mu^{k-1})\right) A_{n-2} \left(\prod_{k=n}^{n} O'_{2k,2n-2}(-\mu^{k-1})\right).$$

Then, we obtain

$$\det(A) = \det(A_{n-1}) = (-1)^n abq^{2(n-1)^2} \left[(aq - b\mu)(bq - a\mu)\right]^{n-1},$$

as desired. □

Theorem 5. *Let $\{\alpha_i\}_{i=1}^{\infty}$ be a 2-periodic sequence. Let $(A_{i,j})_{1\le i,j\le 2n}$ be the doubly indexed sequence given by the recurrences*

$$A_{i,j} = \begin{cases} A_{i-2,j} + A_{i-1,j+1} + A_{i,j+2}, & \text{if} \quad i \ge j+2; \\ A_{i,j-2} + A_{i+1,j-1} + A_{i+2,j}, & \text{if} \quad j \ge i+2, \end{cases}$$

and the initial conditions $A_{i,i} = 0$ for $1 \le i \le 2n$ and $A_{i,i+1} = -A_{i+1,i} = \alpha_i$ for $1 \le i \le 2n-1$. Then we have

$$\det(A_{i,j})_{1\le i,j\le 2n} = \left\{2^{n-3}(\alpha_1 + \alpha_2)^{n-1}\left[4\alpha_1 + (n-1)(n-4)(\alpha_1 - \alpha_2)\right]\right\}^2.$$

Proof. If $n \le 4$, the result is straightforward. Therefore, we may assume that $n \ge 5$. Let A denote the matrix $(A_{i,j})_{1\le i,j\le 2n}$. Now, we apply the row and column operations. Like the proof of Theorem 2, we change the rows and columns such that the columns with nonzero first entries are put together and then the rows with zero first entries are beside one another, and finally we deduce that

$$\det(A) = \begin{cases} \det \begin{bmatrix} T_n & 0 \\ 0 & -(T_n)^T \end{bmatrix}, & n \text{ is even;} \\ -\det \begin{bmatrix} T_n & 0 \\ 0 & -(T_n)^T \end{bmatrix}, & n \text{ is odd,} \end{cases}$$

or equivalently

$$\det(A) = \det \begin{bmatrix} T_n & 0 \\ 0 & (T_n)^T \end{bmatrix} = \det(T_n)^2,$$

where $T_n = (t_{i,j})_{1\le i,j\le n}$ is an $n \times n$ Toeplitz matrix with entries:

$$\begin{cases} t_{11} = \alpha_1, & t_{1,j} = 3t_{1,j-1} + \alpha_2 - \alpha_1 \quad \text{if} \quad j > 1; \\ t_{21} = -\alpha_2, & t_{i,1} = 3t_{i-1,1} + \alpha_2 - \alpha_1 \quad \text{if} \quad i > 2. \end{cases}$$

For instance, when $n = 5$, the matrix T_5 is given by

$$T_5 = \begin{bmatrix} \alpha_1 & 2\alpha_1+\alpha_2 & 5\alpha_1+4\alpha_2 & 14\alpha_1+13\alpha_2 & 41\alpha_1+40\alpha_2 \\ -\alpha_2 & \alpha_1 & 2\alpha_1+\alpha_2 & 5\alpha_1+4\alpha_2 & 14\alpha_1+13\alpha_2 \\ -\alpha_1-2\alpha_2 & -\alpha_2 & \alpha_1 & 2\alpha_1+\alpha_2 & 5\alpha_1+4\alpha_2 \\ -4\alpha_1-5\alpha_2 & -\alpha_1-2\alpha_2 & -\alpha_2 & \alpha_1 & 2\alpha_1+\alpha_2 \\ -13\alpha_1-14\alpha_2 & -4\alpha_1-5\alpha_2 & -\alpha_1-2\alpha_2 & -\alpha_2 & \alpha_1 \end{bmatrix}.$$

Now, it is enough to show that

$$\det(T_n) = 2^{n-3}(\alpha_1+\alpha_2)^{n-1}\big[4\alpha_1 + (n-1)(n-4)(\alpha_1-\alpha_2)\big]. \tag{2}$$

To do this, we apply the following row and column operations:

Step 1. $\mathrm{C}_i \longrightarrow \mathrm{C}_i - 4\mathrm{C}_{i-1} + 3\mathrm{C}_{i-2}$, for $i = n, n-1, \ldots, 3$;
Step 2. $\mathrm{C}_2 \longrightarrow \mathrm{C}_2 - \mathrm{C}_1$;
Step 3. Take the factor $\alpha_1+\alpha_2$ out from columns $\mathrm{C}_2, \mathrm{C}_3, \ldots, \mathrm{C}_n$;
Step 4. $\mathrm{R}_2 \longrightarrow \mathrm{R}_2 + \mathrm{R}_1$;
Step 5. Take the factor 2 out from columns $\mathrm{C}_3, \mathrm{C}_4, \ldots, \mathrm{C}_n$;
Step 6. $\mathrm{R}_i \longrightarrow \mathrm{R}_i - 3\mathrm{R}_{i-1}$, for $i = n, n-1, \ldots, 2$;
Step 7. $\mathrm{R}_i \longrightarrow \mathrm{R}_i - \dfrac{a_{i,i+1}}{a_{i+1,i+1}}\mathrm{R}_{i+1}$, for $i = n-1, n-2, \ldots, 2$;

Therefore, we obtain

$$\det(T_n) = 2^{n-2}(\alpha_1+\alpha_2)^{n-1} \times \det(T^*),$$

where T^* is an $n \times n$ lower triangular matrix with diagonal entries:

$$\frac{4\alpha_1 + (\alpha_1-\alpha_2)(n-1)(n-4)}{2[5-6(n-1)]}, \frac{5-6(n-1)}{n-1}, \frac{n-1}{n-2}, \frac{n-2}{n-3}, \ldots, \frac{i}{i-1}, \ldots, \frac{3}{2}, \frac{2}{1},$$

and so we deduce the validity of Eq. (2). □

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